Oxford Cambridge and RSA

# Monday 05 October 2020 – Afternoon

## A Level Further Mathematics B (MEI)

**Y420/01** Core Pure

**Time allowed: 2 hours 40 minutes**

**You must have:**
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

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**INSTRUCTIONS**
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

**INFORMATION**
- The total mark for this paper is **144**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

**ADVICE**
- Read each question carefully before you start your answer.

**Section A** (36 marks)

Answer **all** the questions.

1 Using standard summation of series formulae, determine the sum of the first $n$ terms of the series

$(1\times2\times4)+(2\times3\times5)+(3\times4\times6)+\ldots,$

where $n$ is a positive integer. Give your answer in fully factorised form. **[6]**

$r(r+1)(r+3)$
$= r(r^2+3r+r+3)$
$= r^3+4r^2+3r$

$\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$

$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$

$\sum_{r=1}^{n} r(r+1)(r+3)$

$= \sum_{r=1}^{n} r^3 + 4\sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r$

$= \frac{n^2}{4}(n+1)^2 + \frac{2n}{3}(n+1)(2n+1) + \frac{3n}{2}(n+1)$

$= \frac{n}{12}(n+1)[3n(n+1) + 8(2n+1) + 18]$

$= \frac{n}{12}(n+1)[3n^2+3n+16n+8+18]$

$= \frac{n}{12}(n+1)(3n^2+19n+26)$

$= \boxed{\frac{n}{12}(n+1)(3n+13)(n+2)}$

**2** **(a)** The matrices $\mathbf{M} = \begin{pmatrix} 0 & 1 & a \\ 1 & b & 0 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} b & -5 \\ -1 & c \\ -1 & 1 \end{pmatrix}$ are such that $\mathbf{MN} = \mathbf{I}$.

Find $a$, $b$ and $c$. **[5]**

**(b)** State with a reason whether or not $\mathbf{N}$ is the inverse of $\mathbf{M}$. **[1]**

(a.) $\underline{\underline{M}}\,\underline{\underline{N}} = \underline{\underline{I}}$

$$\underset{[2\times 3]}{\begin{pmatrix} 0 & 1 & a \\ 1 & b & 0 \end{pmatrix}} \underset{[3\times 2]}{\begin{pmatrix} b & -5 \\ -1 & c \\ -1 & 1 \end{pmatrix}} = \underset{[2\times 2]}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$0b + 1(-1) + a(-1) = 1 \Rightarrow -1 - a = 1 \Rightarrow \therefore a = -1-1 = -2$

$0(-5) + 1c + a(1) = 0 \Rightarrow c + a = 0 \Rightarrow \therefore c = -a = 2$

$1b + b(-1) + 0(-1) = 0 \Rightarrow b - b = 0$

$1(-5) + bc + 0(1) = 1 \Rightarrow -5 + bc = 1 \Rightarrow \therefore b = \frac{1+5}{c} = 3$

$\boxed{\therefore a = -2,\ b = 3,\ c = 2}$

(b.) $\underline{\underline{M}}$ is not a square matrix, so it has no inverse.

**3 In this question you must show detailed reasoning.**

Find $\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}}\,dx$, expressing your answer in terms of $\pi$. **[4]**

$$\int_0^{\frac{1}{3}} \frac{1}{\sqrt{9\left(\frac{4}{9}-x^2\right)}}\,dx = \frac{1}{3}\int_0^{\frac{1}{3}} \frac{1}{\sqrt{\frac{4}{9}-x^2}}\,dx$$

$$= \left[\frac{1}{3}\arcsin\left(\frac{3x}{2}\right)\right]_0^{\frac{1}{3}}$$

$$= \frac{1}{3}\left[\arcsin\left(\frac{1}{2}\right) - \arcsin(0)\right]$$

$$= \frac{1}{3}\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{18}$$

$$\boxed{\therefore \int_0^{\frac{1}{3}} \frac{1}{\sqrt{4-9x^2}}\,dx = \frac{\pi}{18}}$$

+ (-) + -

4 The roots of the equation $2x^3 - 5x + 7 = 0$ are $\alpha$, $\beta$ and $\gamma$.

(a) Find $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$. [4]

(b) Find an equation with integer coefficients whose roots are $2\alpha - 1$, $2\beta - 1$ and $2\gamma - 1$. [4]

(a.) $\alpha + \beta + \gamma = 0$ (true but not necessary to solve this problem)

$\beta\gamma + \alpha\gamma + \alpha\beta = -\frac{5}{2}$

$\alpha\beta\gamma = -\frac{7}{2}$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{\left(-\frac{5}{2}\right)}{\left(-\frac{7}{2}\right)} = \frac{5}{7}$$

$$\boxed{\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{5}{7}}$$

(b.) Roots $= 2\alpha - 1, 2\beta - 1, 2\gamma - 1$

Let $w = 2x - 1$, $\therefore x = \frac{w+1}{2}$

$$2\left(\frac{w+1}{2}\right)^3 - 5\left(\frac{w+1}{2}\right) + 7 = 0$$

$$\frac{1}{\cancel{4}}(w+1)^3 - \frac{5}{2}(w+1) + 7 = 0$$

$\times \cancel{4}$ $\quad \times 4$ $\quad \times 4$

$$(w+1)^3 - 10(w+1) + 28 = 0$$

$$w^3 + 3w^2 + 3w + 1 - 10w - 10 + 28 = 0$$

$$\boxed{\therefore w^3 + 3w^2 - 7w + 19 = 0}$$

**5** Fig. 5 shows the curve with polar equation $r = a(3+2\cos\theta)$ for $-\pi \leqslant \theta \leqslant \pi$, where $a$ is a constant.

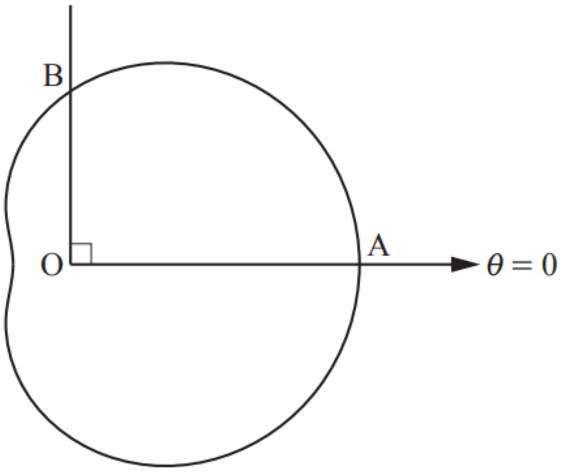


**Fig. 5**

**(a)** Write down the polar coordinates of the points A and B. **[2]**

**(b)** Explain why the curve is symmetrical about the initial line. **[2]**

**(c)** **In this question you must show detailed reasoning.**

Find in terms of $a$ the exact area of the region enclosed by the curve. **[4]**

(a) $\theta = 0 \Rightarrow r = a(3+2\cos 0) = a(3+2) = 5a \Rightarrow \boxed{\therefore A = [5a, 0]}$

$\theta = \frac{\pi}{2} \Rightarrow r = a\left(3+2\cos\frac{\pi}{2}\right) = 3a \Rightarrow \boxed{\therefore B = \left[3a, \frac{\pi}{2}\right]}$

(b) $\boxed{\cos(-\theta) = \cos\theta}$

So the value of r for $-\theta$ is the same as $\theta$.

(c.) $A = \frac{1}{2}\int r^2 d\theta$

$$r^2 = [a(3+2\cos\theta)]^2$$
$$= a^2(3+2\cos\theta)^2$$
$$= a^2(9 + 6\cos\theta + 6\cos\theta + 4\cos^2\theta)$$
$$= a^2(9 + 12\cos\theta + 4\cos^2\theta)$$

N.B
Limits could have also been $2\pi$ to 0

$$A = \frac{1}{2}a^2 \int_{-\pi}^{\pi} 9 + 12\cos\theta + 4\cos^2\theta \; d\theta$$
$$= \frac{1}{2}a^2 \int_{-\pi}^{\pi} 9 + 12\cos\theta + 4\left(\frac{\cos 2\theta + 1}{2}\right) d\theta$$

since $\cos 2\theta = 2\cos^2\theta - 1$

$$= \frac{1}{2}a^2 \int_{-\pi}^{\pi} 11 + 12\cos\theta + 2\cos 2\theta \; d\theta$$
$$= \frac{1}{2}a^2 \left[11\theta + 12\sin\theta + \sin 2\theta\right]_{-\pi}^{\pi}$$
$$= \frac{1}{2}a^2 \left[(11\pi + 0 + 0) - (-11\pi + 0 + 0)\right]$$
$$= \frac{1}{2}a^2(22\pi)$$
$$= 11\pi a^2$$

$$\boxed{\therefore A = 11\pi a^2}$$

**6** The complex number $z$ satisfies the equation $z^2 - 4iz^* + 11 = 0$.

Given that $\mathrm{Re}(z) > 0$, find $z$ in the form $a + bi$, where $a$ and $b$ are real numbers. **[4]**

$z = a + bi$

$z^2 = (a+bi)^2 = a^2 - b^2 + 2abi$

$z^* = a - bi$

$z^2 - 4iz^* + 11 = 0$

$a^2 - b^2 + 2abi - 4i(a - bi) + 11 = 0 \ (+ 0i)$

$a^2 - b^2 + 2abi - 4ai - 4b + 11 = 0$

Real Part : $a^2 - b^2 - 4b + 11 = 0$

Im Part : $2\cancel{a}b - 4\cancel{a} = 0 \Rightarrow \therefore b = \frac{4}{2} = 2$

$\div \cancel{a} \quad \div \cancel{a}$

$a^2 - (2)^2 - 4(2) + 11 = 0$

$a^2 - 4 - 8 + 11 = 0$

$a^2 = 1$

$\therefore a = 1$

$\boxed{\therefore z = 1 + 2i}$

**Section B** (108 marks)

Answer **all** the questions.

7 Prove by mathematical induction that $\sum_{r=1}^{n}(r \times r!) = (n+1)! - 1$ for all positive integers $n$. **[6]**

① Let $n=1$:

$$\text{LHS} = \sum_{r=1}^{1} r \times r! = 1 \times 1! = 1$$

$$\text{RHS} = (n+1)! - 1 = (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$$

LHS = RHS $\therefore$ True for $n=1$.

② Assume true for $n=k$: $\sum_{r=1}^{k} r \times r! = (k+1)! - 1$

③ Let $n=k+1$:

$$\sum_{r=1}^{k+1} r \times r! = \sum_{r=1}^{k} r \times r! + (k+1) \times (k+1)!$$

$$= (k+1)! - 1 + (k+1) \times (k+1)!$$

$$= (k+1)!(1+k+1) - 1$$

$$= (k+2)! - 1$$

$$= (k+1+1)! - 1$$

$\therefore$ True for $n=k+1$, if true for $n=k$.

④ So if true for $n=k$, then true for $n=k+1$ & true for $n=1$, $\therefore$ true for all positive $n$.

**8** **(a)** Given that the lines $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ k \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ meet, determine $k$. **[5]**

**(b)** **In this question you must show detailed reasoning.**

Find the acute angle between the two lines. **[4]**

(a.) $-\lambda = -1 + 2\mu \Rightarrow \lambda = 1 - 2\mu$

$\cancel{2} + \lambda = \cancel{2} + 3\mu \Rightarrow 1 - 2\mu = 3\mu \Rightarrow 1 = 5\mu \Rightarrow \therefore \mu = \frac{1}{5}$

$2 + 3\lambda = k + 4\mu \Rightarrow k = 2 + 3\lambda - 4\mu$

$\mu = \frac{1}{5} \Rightarrow \therefore \lambda = 1 - 2\left(\frac{1}{5}\right) = \frac{3}{5}$

$k = 2 + 3\left(\frac{3}{5}\right) - 4\left(\frac{1}{5}\right) = 3$

$\boxed{\therefore k = 3}$

(b.) $\cos\theta = \dfrac{\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}}{\sqrt{(-1)^2 + 1^2 + 3^2}\sqrt{2^2 + 3^2 + 4^2}}$

as $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|a||b|}$

$\cos\theta = \dfrac{(-1)(2) + (1)(3) + (3)(4)}{\sqrt{11}\sqrt{29}} = \dfrac{13}{\sqrt{319}}$

$\therefore \theta = \cos^{-1}\left(\frac{13}{\sqrt{319}}\right) = 43.292\ldots \approx 43.3^\circ$ (3sf)

$\boxed{\therefore \theta = 43.3^\circ}$

9 A linear transformation of the plane is represented by the matrix $\mathbf{M} = \begin{pmatrix} 1 & -2 \\ \lambda & 3 \end{pmatrix}$, where $\lambda$ is a constant.

**(a)** Find the set of values of $\lambda$ for which the linear transformation has no invariant lines through the origin. **[5]**

**(b)** Given that the transformation multiplies areas by 5 and reverses orientation, find the invariant lines. **[3]**

(a) $\begin{pmatrix} 1 & -2 \\ \lambda & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x-2y \\ \lambda x+3y \end{pmatrix}$

Suppose $y = mx$ is invariant.

$\lambda x + 3y = m(x-2y)$

$\lambda \cancel{x} + 3m\cancel{x} = m(\cancel{x} - 2m\cancel{x})$

$\div x \qquad \div x$

$\lambda + 3m = m(1-2m)$

$\lambda + 3m = m - 2m^2$

$\therefore 2m^2 + 2m + \lambda = 0$

No solutions if discriminant $< 0$.

$b^2 - 4ac = 2^2 - 4(2)(\lambda) = 4 - 8\lambda < 0$

$4 < 8\lambda$

$\lambda > \frac{4}{8}$

$\boxed{\therefore \lambda > \frac{1}{2}}$

(b) $\det \underset{\sim}{M} = 3 + 2\lambda$

Given description $\det \underset{\sim}{M} = -5$, $\therefore 3 + 2\lambda = -5$.

$\therefore \lambda = \frac{-5-3}{2} = -4$

$2m^2 + 2m - 4 = 0 \Rightarrow m^2 + m - 2 = 0 \Rightarrow (m+2)(m-1) = 0$

$\therefore m = -2, 1$ $\quad$ $\boxed{\therefore \text{Lines are: } y = -2x \ \& \ y = x}$

**10 In this question you must show detailed reasoning.**

The region in the first quadrant bounded by curve $y = \cosh\frac{1}{2}x^2$, the $y$-axis, and the line $y = 2$ is rotated through 360° about the $y$-axis.

Find the exact volume of revolution generated, expressing your answer in a form involving a logarithm. **[7]**

About y-axis: $V = \int_a^b \pi x^2 \, dy$

$y = \cosh\frac{1}{2}x^2 \Rightarrow x^2 = 2\cosh^{-1}y$

$V = \int_1^2 2\pi \cosh^{-1}y \, dy$

$= \left[2\pi y\cosh^{-1}y\right]_1^2 - \int_1^2 \frac{2\pi y}{\sqrt{y^2-1}} dy$

① ②

Integration by parts:

$\int uv' dx = uv - \int vu' dx + c$

Let: $u = 2\pi\cosh^{-1}y$ $\quad v = y$

$u' = \frac{2\pi}{\sqrt{y^2-1}}$ $\quad v' = 1$

① $\left[2\pi y\cosh^{-1}y\right]_1^2 = 2\pi(2)\cosh^{-1}2 - 0$

$= 4\pi \ln|2 + \sqrt{2^2-1}|$

$= 4\pi \ln|2+\sqrt{3}|$

② $\int_1^2 \frac{2\pi y}{\sqrt{y^2-1}} dy = 2\pi \int_1^2 y(y^2-1)^{-\frac{1}{2}} dy = 2\pi\left[\sqrt{y^2-1}\right]_1^2$

$= 2\pi\left[\sqrt{2^2-1} - \sqrt{1^2-1}\right]$

$= 2\pi\sqrt{3}$

Let $u = (y^2-1)^{\frac{1}{2}}$

$\frac{du}{dy} = \frac{1}{2}(2y)(y^2-1)^{-\frac{1}{2}} = \frac{y}{\sqrt{y^2-1}}$

$\boxed{\therefore V = 4\pi \ln|2+\sqrt{3}| - 2\pi\sqrt{3}}$

**11 In this question you must show detailed reasoning.**

In Fig. 11, the points A, B, C, D, E and F represent the complex sixth roots of 64 on an Argand diagram. The midpoints of AB, BC, CD, DE, EF and FA are G, H, I, J, K and L respectively.

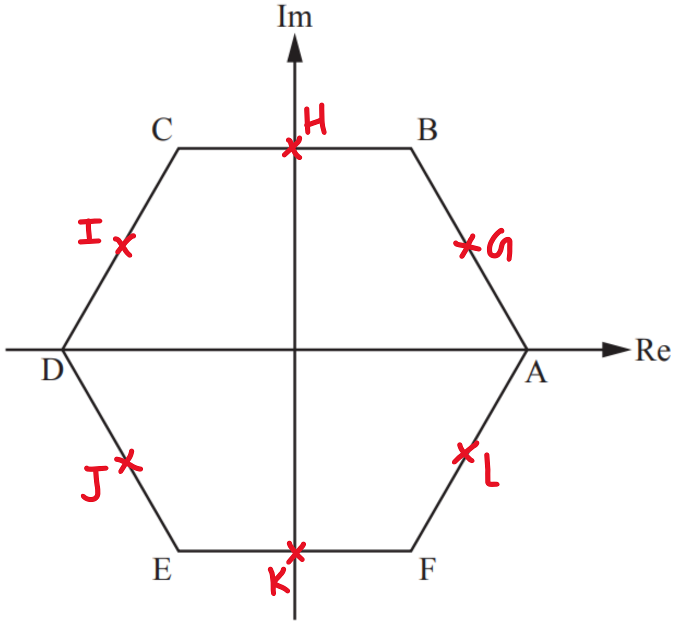


**Fig. 11**

**(a)** Write down, in exponential ($re^{i\theta}$) form, the complex numbers represented by the points A, B, C, D, E and F. **[2]**

**(b)** When these complex numbers are multiplied by the complex number $w$, the resulting complex numbers are represented by the points G, H, I, J, K and L.

Find $w$ in exponential form. **[4]**

**(c)** You are given that G, H, I, J, K and L represent roots of the equation $z^6 = p$.

Find $p$. **[2]**

(a.) $A = 2$, $B = 2e^{\frac{i\pi}{3}}$, $C = 2e^{\frac{2i\pi}{3}}$, $D = -2$

$E = 2e^{\frac{4i\pi}{3}}$, $F = 2e^{\frac{5i\pi}{3}}$

(b.) Modulus of $G = \sqrt{3}$ $\Rightarrow \therefore$ Modulus of $w = \frac{\sqrt{3}}{2}$

Argument of $w = \frac{\pi}{3} \div 2 = \frac{\pi}{6}$ $\boxed{\therefore w = \frac{\sqrt{3}}{2} e^{\frac{i\pi}{6}}}$

(c.) $z^6 = p$

$\left(\sqrt{3} e^{\frac{i\pi}{6}}\right)^6 = (\sqrt{3})^6 e^{i\pi} = 27e^{i\pi} = -27$ $\boxed{\therefore p = -27}$

**12** **(a)** Given that $z = \cos\theta + \mathrm{i}\sin\theta$, express $z^n + \frac{1}{z^n}$ and $z^n - \frac{1}{z^n}$ in simplified trigonometric form. **[2]**

**(b)** By considering $\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3$, find constants $A$ and $B$ such that

$\sin^3\theta\cos^3\theta = A\sin 6\theta + B\sin 2\theta$. **[6]**

(a.)

$$\boxed{\begin{aligned} z^n + \frac{1}{z^n} &= 2\cos n\theta \\ z^n - \frac{1}{z^n} &= 2i\sin n\theta \end{aligned}}$$

(b.) $\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3 = (2\cos\theta)^3(2i\sin\theta)^3 = -64i\cos^3\theta\sin^3\theta$

$$\left(z+\frac{1}{z}\right)^3\left(z-\frac{1}{z}\right)^3 = \left(z^3+3z+\frac{3}{z}+\frac{1}{z^3}\right)\left(z^3-3z+\frac{3}{z}-\frac{1}{z^3}\right)$$

$$= z^6 - \frac{1}{z^6} - 3z^2 + \frac{3}{z^2}$$

From de moivre's theorem →

$$= 2i\sin 6\theta - 3(2i\sin 2\theta)$$

$$= 2i\sin 6\theta - 6i\sin 2\theta$$

$$-64i\cos^3\theta\sin^3\theta = 2i\sin 6\theta - 6i\sin 2\theta$$

$$\cos^3\theta\sin^3\theta = \frac{2i}{-64i}\sin 6\theta - \frac{6i}{-64i}\sin 2\theta$$

$$\boxed{\begin{aligned} &\therefore \cos^3\theta\sin^3\theta = -\frac{1}{32}\sin 6\theta + \frac{3}{32}\sin 2\theta \\ &\therefore A = -\frac{1}{32},\ B = \frac{3}{32} \end{aligned}}$$

**13** **(a)** Using exponentials, prove that $\sinh 2x = 2\cosh x \sinh x$. **[2]**

**(b)** Hence show that if $f(x) = \sinh^2 x$, then $f''(x) = 2\cosh 2x$. **[2]**

**(c)** Explain why the coefficients of odd powers in the Maclaurin series for $\sinh^2 x$ are all zero. **[2]**

**(d)** Find the coefficient of $x^n$ in this series when $n$ is a positive even number. **[3]**

(a.) RHS $= 2\cosh x \sinh x$

$$= 2\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)$$

$$= \frac{1}{2}(e^x+e^{-x})(e^x-e^{-x})$$

$$= \frac{e^{2x}-e^{-2x}}{2}$$

$$= \sinh 2x = \text{LHS}$$

$\boxed{\therefore \sinh 2x = 2\cosh x \sinh x}$

(b.) $f(x) = \sinh^2 x$

$f'(x) = 2\sinh x \cosh x = \sinh 2x$

$\boxed{\therefore f''(x) = 2\cosh 2x}$

$u = \sinh x \Rightarrow u' = \cosh x$

$y = 2x \Rightarrow \frac{dy}{dx} = 2$

(c.) $f'''(x) = 4\sin 2x$

$f^5(x) = 16\sinh 2x$

...

$\therefore$ Odd derivatives are multiples of $\sinh 2x$.

$\therefore f'''(0) = f^5(0) = f^7(0) = \ldots = 0$

(d.) $f''(x) = 2\cosh 2x$

$f^4(x) = 8\cosh 2x$

...

$\therefore f^n(0) = 2^{n-1}$ [n even]

$\boxed{\therefore \text{Coefficient of } x^n = \frac{2^{n-1}}{n!}}$

**14** Solve the simultaneous differential equations

① $\frac{dx}{dt} + 2x = 4y$, ② $\frac{dy}{dt} + 3x = 5y$, differentiate both equations

given that when $t = 0$, $x = 0$ and $y = 1$. **[11]**

① $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 4\frac{dy}{dt} = 4(5y - 3x) = 20y - 12x$

② $\frac{dy}{dt} = 5y - 3x$ Substitute

From ① $y = \frac{1}{4}\frac{dx}{dt} + \frac{x}{2} \Rightarrow \frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 20\left(\frac{1}{4}\frac{dx}{dt} + \frac{x}{2}\right) - 12x$

$\therefore \frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$

Auxillary Equation: $\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

$\therefore \lambda = 1, \lambda = 2$

General Solution: $x = Ae^t + Be^{2t} \Rightarrow \therefore \frac{dx}{dt} = Ae^t + 2Be^{2t}$

$y = \frac{1}{4}\frac{dx}{dt} + \frac{x}{2} = \frac{1}{4}(Ae^t + 2Be^{2t}) + \frac{Ae^t}{2} + \frac{Be^{2t}}{2} = \frac{3}{4}Ae^t + Be^{2t}$

$t = 0, x = 0, y = 1 \Rightarrow y = 1 = \frac{3}{4}A + B$ $\times 4$ $\times 4$ $\times 4$

$\therefore 3A + 4B = 4$ ①

$x = 0 = A + B$

$\therefore A + B = 0$ ②

① $\cancel{3A} + 4B = 4$

$3\times$② $\cancel{3A} + 3B = 0$

$\therefore B = 4 \Rightarrow \therefore A = -B = -4$

$$\boxed{\begin{aligned} &\therefore x = 4e^{2t} - 4e^t \\ &\therefore y = 4e^{2t} - 3e^t \end{aligned}}$$

**15** **(a)** Show that the three planes with equations

$$x+\lambda y+3z=-12$$
$$2x+\ \ y+5z=-11$$
$$x-2y+2z=-9$$

where $\lambda$ is a constant, meet at a unique point except for one value of $\lambda$ which is to be determined. **[3]**

**(b)** In the case $\lambda=-2$, use matrices to find the point of intersection P of the planes, showing your method clearly. **[3]**

The line $l$ has equation $\dfrac{x-1}{2}=\dfrac{y-1}{-1}=\dfrac{z+2}{-2}$.

**(c)** Find a vector equation of $l$. **[2]**

**(d)** Find the shortest distance between the point P and $l$. **[4]**

**(e)** **(i)** Show that $l$ is parallel to the plane $x-2y+2z=-9$. **[3]**

**(ii)** Find the distance between $l$ and the plane $x-2y+2z=-9$. **[2]**

(a.) $\begin{array}{ccc} + & - & + \end{array}$

$$\begin{vmatrix} 1 & \lambda & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 5 \\ -2 & 2 \end{vmatrix} - \lambda\begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 1(2+10) - \lambda(4-5) + 3(-4-1)$$

$$= 12 + \lambda - 15$$

$$= \lambda - 3$$

When determinant $=0$, matrix is singular, $\therefore$ no inverse.

$\therefore$ A single, determined PoI doesn't exist when det $=0$.

$\lambda - 3 = 0 \Rightarrow \boxed{\therefore \lambda = 3}$

(b.) $\lambda=-2 \Rightarrow \underline{M} = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 2 \end{pmatrix} \Rightarrow$ det $\underline{M} = -5$ from the equation in part (a)

$$\underline{M}^{-1} = \frac{1}{5}\begin{pmatrix} -12 & 2 & 13 \\ -1 & 1 & -1 \\ 5 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{M}^{-1}\begin{pmatrix} -12 \\ -11 \\ -9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \qquad \boxed{\therefore x=1,\ y=2,\ z=-3}$$

(c.) $\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z+2}{-2} = \lambda$

$\therefore x = 2\lambda+1,\ y = -\lambda+1,\ z = -2\lambda-2$

$$\boxed{\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}}$$

(d.) $\overrightarrow{AP} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\overrightarrow{AP} \times \underline{u} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2-1 \\ -2-0 \\ 0-2 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -2 \end{pmatrix}$

$d = \frac{|\overrightarrow{AP} \times \underline{u}|}{|\underline{u}|} = \frac{\sqrt{(-3)^2+(-2)^2+(-2)^2}}{\sqrt{2^2+(-1)^2+(-2)^2}} = \frac{\sqrt{17}}{3}$

$$\boxed{\therefore d = \frac{\sqrt{17}}{3}}$$

(e.) (i) $x - 2y + 2z = -9 \Rightarrow \therefore \underline{n} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$\underline{n} \cdot \underline{u} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = (1)(2) + (-2)(-1) + (2)(-2) = 0$

Since $\underline{n} \cdot \underline{u} = 0$, line $L$ is parallel to the plane.

(ii) Distance between $(1, 1, -2)$ & $\overset{1}{x} \overset{-2}{-2y} \overset{+2}{+2z} = -9$ :

$d = \frac{|1(1) + 1(-2) - 2(2) + 9|}{\sqrt{1^2 + (-2)^2 + 2^2}} = \frac{4}{3}$

$$\boxed{\therefore d = \frac{4}{3}}$$

**16** The population density $P$, in suitable units, of a certain bacterium at time $t$ hours is to be modelled by a differential equation. Initially, the population density is zero, and its long-term value is $A$.

**(a)** One simple model is to assume that the rate of change of population density is directly proportional to $A - P$.

**(i)** Formulate a differential equation for this model. **[1]**

**(ii)** Verify that $P = A(1 - e^{-kt})$, where $k$ is a positive constant, satisfies

- this differential equation,
- the initial condition,
- the long-term condition. **[3]**

An alternative model uses the differential equation

$$\frac{dP}{dt} - \frac{P}{t(1+t^2)} = Q(t),$$

where $Q(t)$ is a function of $t$.

**(b)** Find the integrating factor for this differential equation, showing that it can be written in the form $\frac{\sqrt{1+t^2}}{t}$. **[8]**

**(c)** Suppose that $Q(t) = 0$.

**(i)** Show that $P = \frac{At}{\sqrt{1+t^2}}$. **[4]**

**(ii)** Find the time predicted by this model for the population density to reach half its long-term value. Give your answer correct to the nearest minute. **[2]**

**(d)** Now suppose that $Q(t) = \frac{te^{-t}}{\sqrt{1+t^2}}$.

Show that $P = \frac{At - te^{-t}}{\sqrt{1+t^2}}$. [You may assume that $\lim_{t \to \infty} te^{-t} = 0$.] **[5]**

It is found that the long-term value of $P$ is 10, and $P$ reaches half this value after 37 minutes.

**(e)** Determine which of the models proposed in parts **(c)** and **(d)** is more consistent with these data. **[2]**

(a.) (i) $\frac{dP}{dt} \propto (A-P) \Rightarrow \boxed{\therefore \frac{dP}{dt} = k(A-P)}$

(ii) $P = A(1-e^{-kt})$

$\frac{dP}{dt} = Ake^{-kt} = k(A-P)$ $\quad\therefore$ Correct differential eq.

$t=0 \Rightarrow P = A(1-e^{-k(0)}) = A(1-1) = 0$ $\quad\therefore$ Initial condition met.

As $t \to \infty$, $e^{-kt} \to 0$, so $P \to A$. $\quad\therefore$ Long-term condition met.

(b.) $\frac{dP}{dt} - \frac{P}{t(1+t^2)} = Q(t)$

$\boxed{IF = e^{-\int \frac{1}{t(1+t^2)}\,dt} =}$

$\frac{1}{t(1+t^2)} \equiv \frac{A}{t} + \frac{Bt+C}{1+t^2}$

$A(1+t^2) + t(Bt+C) \equiv 1 \Rightarrow A + At^2 + Bt^2 + Ct \equiv 1$

$t^2$: $A + B = 0$

$t$: $C = 0$

$\_$: $A = 1 \Rightarrow \therefore B = -A = -1$

$\therefore \frac{1}{t(1+t^2)} = \frac{1}{t} - \frac{1}{1+t^2}$

$IF = e^{\int -\frac{1}{t} + \frac{1}{1+t^2}\,dt}$

$= e^{-\ln t + \frac{1}{2}\ln|1+t^2|}$

$= \cancel{e}^{\cancel{\ln}}\left|\frac{\sqrt{1+t^2}}{t}\right|$

$= \frac{\sqrt{1+t^2}}{t}$

$\boxed{\therefore IF = \frac{\sqrt{1+t^2}}{t}}$

(c.)(i) $Q(t)=0 \Rightarrow \frac{d}{dt}\left[P\frac{\sqrt{1+t^2}}{t}\right]=0$

$$P\frac{\sqrt{1+t^2}}{t}=k$$

$$P=\frac{kt}{\sqrt{1+t^2}}$$

$$\lim_{t\to\infty} P = k \qquad \therefore k=A \qquad \boxed{\therefore P=\frac{At}{\sqrt{1+t^2}}}$$

(ii) $\frac{1}{2}\cancel{A} = \frac{\cancel{A}t}{\sqrt{1+t^2}}$

$$\sqrt{1+t^2} = 2t$$

$$\cancel{\square^2} \qquad \square^2$$

$$1+t^2=4t^2$$

$$3t^2=1$$

$$t^2=\frac{1}{3}$$

$$\boxed{\therefore t=\sqrt{\frac{1}{3}} \text{ hours } = 34.64\ldots \text{ mins} \approx 35 \text{ mins}}$$

(d.) $Q(t)=\frac{te^{-t}}{\sqrt{1+t^2}}$

$$\frac{d}{dt}\left[P\frac{\sqrt{1+t^2}}{t}\right]=\frac{\sqrt{1+t^2}}{t}\,\frac{te^{-t}}{\sqrt{1+t^2}}=e^{-t}$$

$$P\frac{\sqrt{1+t^2}}{t}=c-e^{-t}$$

$$\lim_{t\to\infty} P = c = \text{Long-term value of } P$$

$$\therefore c=A \qquad \boxed{\therefore P=\frac{At-te^{-t}}{\sqrt{1+t^2}}}$$

(e.) $A = 10$

By first model, when $t = \frac{37}{60}$, $P = 5.25$.

By second model, when $t = \frac{37}{60}$, $P = 4.97$.

$\therefore$ $2^{nd}$ model fits better.